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On the Higher Relation Modules of a Finite Group

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1. INTRODUCTION

Let R be a normal subgroup of a non-cyclic free group F of finite rank and let $\gamma_k(R)$ ($k \geqslant 1$) denote the k-th term of the lower central series of R. The lower central factors $\gamma_k(R)/\gamma_{k+1}(R)$ can be regarded as right F/R-modules by setting

$$u\gamma_{k+1}(R) \cdot wR = w^{-1}uw\gamma_{k+1}(R), \qquad w \in F, \quad u \in \gamma_k(R).$$

We call $\gamma_k(R)/\gamma_{k+1}(R)$ the k-th relation module of F/R ($R/R' = \gamma_1(R)/\gamma_2(R)$ is the relation module). The purpose of this paper is to study the structure of these higher relation modules and obtain information concerning the group $C_k/\gamma_{k+1}(R)$ which consists of the fixed points of F/R on $\gamma_k(R)/\gamma_{k+1}(R)$. Using a character formula for $\gamma_k(R)/\gamma_{k+1}(R)$ we are able to compute the rank of $C_k/\gamma_{k+1}(R)$. The importance of this result lies in the fact that the group $C_k/\gamma_{k+1}(R)$ is precisely the centre of the group $F/\gamma_{k+1}(R)$ (see for instance [5]).

It is a well-known result of Auslander and Lyndon [1] that F/R' has a non-trivial centre if and only if F/R is finite. This result generalizes to: $F/\gamma_{k+1}(R)$ has a non-trivial centre if and only if F/R is finite (see for instance [5]). Throughout this paper we shall assume that F/R is finite. A result of Gaschütz [3] (see also Ojanguren [9]) shows that the rank of $C_1/\gamma_2(R)$ equals the rank of F. Mittal

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and Passi (1972, verbal communication) have observed that the rank of $C_k/\gamma_{k+1}(R)$ $(k \geqslant 1)$ is not less than the rank of $\gamma_k(F)/\gamma_{k+1}(F)$. Our results show that for $k \geqslant 2$, the rank of $C_k/\gamma_{k+1}(R)$ is much larger than the rank of $\gamma_k(F)/\gamma_{k+1}(F)$ and that its size is influenced by the number of elements x in F/R with $x^k = 1$.

Let G be a finite group given by a free presentation

$$1 \to R \to F \to G \to 1$$

with F of finite rank j exceeding 1. Then our main results may be stated as follows.

THEOREM 4.1. *The character χ_k of the k-th relation module $\gamma_k(R)/\gamma_{k+1}(R)$ $(k \geqslant 1)$ is given by the formula*

$$\chi_k(g) = \frac{1}{k} \sum_{d \mid k} \mu(d)(\chi(g^d))^{k/d}, \qquad g \in G,$$

where

$$\chi(g) = \begin{cases} 1 + |G|(j-1) & \text{if } g = 1 \\ 1 & \text{if } g \neq 1 \end{cases}$$

is the character χ_1 of R/R' and μ is the Möbius function.

THEOREM 5.1. *Let $C_k/\gamma_{k+1}(R)$ $(k \geqslant 1)$ be the centre of $F/\gamma_{k+1}(R)$. Then the rank r_k of $C_k/\gamma_{k+1}(R)$ is given by the formula*

$$r_1 = j, \qquad \text{the rank of } F$$

$$r_k = \frac{1}{k \mid G \mid} \sum_{d \mid k} \mu(d)\, \nu_d(G)(m^{k/d} - 1), \qquad (k \geqslant 2),$$

where $m = 1 + (j-1) \mid G \mid$ = rank of R and $\nu_d(G)$ is the number of elements x in G with $x^d = 1$.

THEOREM 5.2. *If $(|G|, k) = 1$, $k \geqslant 2$, then the rank r_k of $C_k/\gamma_{k+1}(R)$ is $m(k)/|G|$, where $m(k)$ is the rank of $\gamma_k(R)/\gamma_{k+1}(R)$.*

THEOREM 6.3. *If F/R is a non-trivial finite group then, for each $k \geqslant 1$, the second centre of $F/\gamma_{k+1}(R)$ coincides with the centre of $F/\gamma_{k+1}(R)$.*

The paper is organized as follows. In Section 2 we discuss some results from the representation theory of finite groups. In Section 3 we discuss a trace formula due to Brandt [2] for linear transformations on sections of a free Lie algebra. In Section 4 we derive a formula for the character of the module

$\gamma_k(R)/\gamma_{k+1}(R)$ by combining the results of Sections 2 and 3. Here we also discuss the structure of $\gamma_k(R)/\gamma_{k+1}(R) \otimes_{\mathbb{Z}} \mathbb{Q}$ as a $\mathbb{Q}(F/R)$-module. In Section 5 we compute the rank of $C_k/\gamma_{k+1}(R)$ and in Section 6 we record some general properties of the group $F/\gamma_{k+1}(R)$ with F/R finite. Finally, in Section 7 we look at the embedding of a free Lie algebra in the tensor algebra.

Our notation is standard. The relevant material on Lie algebras can be found in Hall [6, Chapter 11] and Serre [10, Chapter 2] and that on group characters in [6, Chapter 16].

2. Representation Theoretic Preliminaries

Let A be a free $\mathbb{Z}$-module of finite rank and G a finite group acting on A as a group of $\mathbb{Z}$-linear transformations. Denote by A_0 the submodule of A consisting of the fixed points of A under the action of G, i.e.,

$$A_0 = \{a \in A \mid a \cdot g = a \text{ for all } g \in G\},$$

and denote by $[A, G]$ the submodule of A given by

$$[A, G] = \{a - a \cdot g \mid a \in A, g \in G\}.$$

Since for $a \in A$,

$$|G| a = a \cdot \sum_{g \in G} g + \sum_{g \in G} (a - a \cdot g),$$

it follows that

$$|G| A \subseteq A_0 + [A, G].$$

We note also that A_0 is a $\mathbb{Z}$-direct summand of A. For, by the fundamental theorem of Abelian groups we can choose a basis $b_1, \ldots, b_m$ of A so that A_0 is freely generated by $h_1 b_1, \ldots, h_m b_m$ for some integers $h_i \geqslant 0$. But $h_i b_i \cdot g = h_i b_i$ implies that $h_i(b_i \cdot g - b_i) = 0$ and so $b_i \cdot g = b_i$ if $h_i \neq 0$. Thus $A = A_0 \oplus A_1$.

Set $V = A \otimes_{\mathbb{Z}} \mathbb{Q}$. Then V is a vector space over $\mathbb{Q}$ with $\dim_{\mathbb{Q}} V = \operatorname{rank} A$. We regard V as a $\mathbb{Q}G$-module in the obvious way. The space of fixed points of G on V is clearly $V_0 = A_0 \otimes_{\mathbb{Z}} \mathbb{Q}$. By Maschke's theorem (see for instance [6, Theorem 16.3.1]) there exists a $\mathbb{Q}G$-submodule V_1 of V such that $V = V_0 \oplus V_1$ (note that V_1 is not necessarily $A_1 \otimes_{\mathbb{Z}} \mathbb{Q}$, since A_1 need not be G-invariant). We next observe that

$$[V, G] = [A, G] \otimes_{\mathbb{Z}} \mathbb{Q}$$

so that

$$V = V_0 + [V, G].$$

On the other hand $V/V_1 \cong V_0$ implies $[V, G] \subseteq V_1$ and we conclude that $V_1 = [V, G]$. Thus

$$V = V_0 \oplus [V, G],$$

and since G has no fixed points on $[V, G]$ we get $[V, G] = [V, G, G]$. We record these observations as

LEMMA 2.1. *$|G| A \subseteq A_0 \oplus [A, G]$ and $|G| [A, G] \subseteq [A, G, G]$.*

Suppose now that χ is the character of the $\mathbb{Q}G$-module V. By Maschke's theorem V is the direct sum of irreducible $\mathbb{Q}G$-modules, $W_1, \ldots, W_\ell$, say. If $W_i \otimes_{\mathbb{Q}} \mathbb{C}$ is not irreducible as a $\mathbb{C}G$-module (this is expressed by saying that W_i is not absolutely irreducible) then it is a direct sum of irreducible complex representations of G none of which is equivalent to a rational representation. Thus $W_i \otimes_{\mathbb{Q}} \mathbb{C}$ can contain the trivial $\mathbb{C}G$-module as a constituent only if W_i is itself trivial. In particular, the dimension of V_0 is the number of times the trivial $\mathbb{C}G$-module arises in a decomposition of $V \otimes_{\mathbb{Q}} \mathbb{C}$ into a direct sum of irreducible $\mathbb{C}G$-modules. The orthogonality relations for irreducible characters ([6], Section 16.6) imply that the number of times an irreducible $\mathbb{C}G$-module W occurs in a direct decomposition of $V \otimes_{\mathbb{Q}} \mathbb{C}$ is

$$\langle \chi, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\phi(g)},$$

where ϕ is the character of W and $\overline{\phi(g)}$ denotes the complex conjugate of $\phi(g)$. In particular, we obtain the following

LEMMA 2.2. *Rank $A_0 = \dim_{\mathbb{Q}} V_0 = \langle \chi, 1_G \rangle = (1/|G|) \sum_{g \in G} \chi(g)$. (Here 1_G is the character of the trivial representation i.e. $1_G(g) = 1$ for all $g \in G$).*

3. LIE THEORETIC PRELIMINARIES

Let L be a free Lie algebra over a field K of characteristic zero, freely generated by $x_1, \ldots, x_n$ $(n < \infty)$. Define $L^1 = L, L^{i+1} = [L^i, L]$ for $i \geqslant 1$. For each $k \geqslant 1$, the homogeneous component $\mathscr{L}_k = L^k/L^{k+1}$ is a vector space over K. A basis for $\mathscr{L}_k$ consists of the Lie basic elements of weight k and the dimension of $\mathscr{L}_k$ is given by the Witt formula

$$\dim \mathscr{L}_k = \frac{1}{k} \sum_{d \mid k} \mu(d)\, n^{k/d},$$

where

$$\mu(d) = \begin{cases} 1 & \text{if } d = 1 \\ 0 & \text{if } d \text{ is divisible by } p^2 \text{ for some prime } p \\ (-1)^r & \text{if } d = p_1 \cdots p_r \text{ is a product of distinct primes} \end{cases}$$

is the Möbius function (see [6], Theorem 11.2.2). If T is a linear transformation of $\mathscr{L}_1$, then T induces in a natural way a linear transformation T_k of $\mathscr{L}_k$. The trace, tr T_k, of T_k has been computed by Brandt [2] in terms of k and tr T (see also Wever [12]). The result is as follows.

LEMMA 3.1 (Brandt [2]). *Let T_k denote the linear transformation of $\mathscr{L}_k$ induced by a linear transformation T of $\mathscr{L}_1$. Then*

$$\operatorname{tr} T_k = \frac{1}{k} \sum_{d|k} \mu(d)(\operatorname{tr} T^d)^{k/d}.$$

A quick derivation of the above trace formula can be outlined as follows. Regarding the formula as an identity connecting the entries of the matrices of T, T_k, it suffices to verify the formula for a matrix T whose entries are distinct commuting indeterminates and K, an algebraically closed field containing these indeterminates. Since such a matrix T is similar to a diagonal matrix over K, we may assume that T is diagonal and that $x_1,..., x_n$ are its eigenvectors with the corresponding eigenvalues $\lambda_1,..., \lambda_n$ (i.e., $Tx_i = \lambda_i x_i$, for all $i = 1,..., n$). Let $(k_1,..., k_n)$ be an n-tuple of non-negative integers with $\sum_i k_i = k$. A basic element $\beta(x_1,..., x_n)$ of $\mathscr{L}_k$ has weight class $(k_1,..., k_n)$ if the weight of x_i in $\beta(x_1,..., x_n)$ is precisely k_i. In this case

$$\begin{aligned} T_k\beta(x_1,..., x_n) &= \beta(Tx_1,..., Tx_n) \\ &= \beta(\lambda_1 x_1,..., \lambda_n x_n) \\ &= \lambda_1^{k_1} \cdots \lambda_n^{k_n} \beta(x_1,..., x_n). \end{aligned}$$

Hence,

$$\begin{aligned} \operatorname{tr} T_k &= \sum_{\beta(x_1,...,x_n)} \lambda_1^{k_1} \cdots \lambda_n^{k_n} \\ &= \sum_{(k_1,...,k_n)} | M(k_1,..., k_n)| \, \lambda_1^{k_1} \cdots \lambda_n^{k_n} \qquad (*) \end{aligned}$$

where $M(k_1,..., k_n)$ is the (possibly empty) set of all basic elements of $\mathscr{L}_k$ with the assigned weight class $(k_1,..., k_n)$. A formula of Witt (see [6], Theorem 11.2.2) gives

$$| M(k_1,..., k_n)| = \frac{1}{k} \sum_{d|\langle k_1,...,k_n\rangle} \mu(d) \frac{(k/d)!}{(k_1/d)! \cdots (k_n/d)!}$$

where $\langle k_1, \ldots, k_n \rangle$ denotes the highest common factor of $k_1, \ldots, k_n$. Substituting into (*) yields

$$\begin{aligned} \operatorname{tr} T_k &= \frac{1}{k} \sum_{d \mid k} \mu(d)(\lambda_1{}^d + \cdots + \lambda_n{}^d)^{k/d} \\ &= \frac{1}{k} \sum_{d \mid k} \mu(d)(\operatorname{tr} T^d)^{k/d}, \end{aligned}$$

as required.

4. The Character of the k-th Relation Module

Let F be a non-cyclic free group and R a normal subgroup of F. Denote by L the graded Lie ring whose underlying abelian group is $\oplus_{k \geqslant 1} \gamma_k(R)/\gamma_{k+1}(R)$ and whose Lie bracket is induced by linearly extending the commutation

$$[u\gamma_{i+1}(R), v\gamma_{j+1}(R)] = u^{-1}v^{-1}uv\gamma_{i+j+1}(R),$$

$u \in \gamma_i(R)$, $v \in \gamma_j(R)$. Since R is a (free) subgroup of F, L is the free Lie algebra (over $\mathbb{Z}$) generated by R/R' (see for instance Serre [10], Chapter 2). The action of $G = F/R$ on the homogeneous components $\gamma_k(R)/\gamma_{k+1}(R)$ (via conjugation) induces non-singular $\mathbb{Z}$-linear transformations of the underlying abelian group of L. These transformations are easily seen to be Lie algebra automorphisms of L and are therefore induced by naturally extending the action of G on R/R' to all of L. In this way the action of $g \in G$ on R/R' gives rise to a $\mathbb{Z}$-linear transformation $T(g)$ on L/L^2. In turn, $T(g)$ induces a $\mathbb{Z}$-linear transformation $T_k(g)$ on $L^k/L^{k+1} = \mathscr{L}_k \cong \gamma_k(R)/\gamma_{k+1}(R)$ compatible with the action of g on $\gamma_k(R)/\gamma_{k+1}(R)$.

If F is free of rank j ($j \geqslant 2$) and $G = F/R$ is finite then L is a free Lie algebra of finite rank $1 + |G|(j-1) = \operatorname{rank} R/R'$. Let χ_k denote the character of the representation of G on $\gamma_k(R)/\gamma_{k+1}(R)$. Then $\chi_k(g) = \operatorname{tr} T_k(g)$. Replacing L by $L \otimes_{\mathbb{Z}} \mathbb{C}$, we may apply Lemma 3.1 to conclude that

$$\begin{aligned} \chi_k(g) &= \frac{1}{k} \sum_{d \mid k} \mu(d)(\operatorname{tr} T(g)^d)^{k/d} \\ &= \frac{1}{k} \sum_{d \mid k} \mu(d)(\operatorname{tr} T(g^d))^{k/d} \end{aligned}$$

We have thus proved the following result.

THEOREM 4.1. *The character* χ_k *of the k-th relation module* $\gamma_k(R)/\gamma_{k+1}(R)$ $(k \geqslant 1)$ *is given by the formula*

$$\chi_k(g) = \frac{1}{k} \sum_{d \mid k} \mu(d)(\chi(g^d))^{k/d}, \qquad (g \in G)$$

where χ *is the character* χ_1 *of* R/R'.

A formula for the character χ of R/R' has been obtained by Gaschütz [3] (cf. Ojanguren [9]). Because of the importance of this formula we give here a simple derivation of it. First note that if $G = F/R$ is cyclic of order $|G| \neq 1$, an elementary argument using a Schreier transversal shows ([9], Satz 5.6) that R/R' has a basis on which a generator g of G is represented by

$$T(g) = (1) \oplus \underbrace{E \oplus \cdots \oplus E}_{\leftarrow j-1 \rightarrow},$$

where $j = \operatorname{rank} F$, (1) denotes the 1×1 identity matrix and E is the $|G| \times |G|$ permutation matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

In particular $T(g^i)$ has trace 1 for $i = 1, 2, \ldots, |G| - 1$. Now in the general case, to compute the character of G on R/R', we may proceed elementwise. If $1 \neq g \in G$ and $H \leqslant F$ with $H/R = \langle g \rangle$, then H is free and H/R is cyclic. We conclude that $\chi(g) = 1$. Thus we have the following theorem.

THEOREM 4.2 (Gaschütz [3]). *The character* χ *of the relation module* R/R' *is given by*

$$\chi(g) = \begin{cases} 1 + |G|(j-1) & \text{if } g = 1 \\ 1 & \text{if } g \neq 1. \end{cases}$$

Since, for a field K of characteristic zero, two KG-modules are equivalent if and only if they have the same character, Theorems 4.1 and 4.2 determine (in principle) the structure of all relation modules $\gamma_k(R)/\gamma_{k+1}(R) \otimes_{\mathbb{Z}} \mathbb{Q}$. Since χ is the sum of the trivial character 1_G and $j - 1$ copies of the regular character ρ_G, it follows that

$$R/R' \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q} \oplus \underbrace{\mathbb{Q}(G) \oplus \cdots \oplus \mathbb{Q}(G)}_{\longleftarrow j-1 \longrightarrow}.$$

Because of the influence on $\chi_k(g)$, $k \geqslant 2$, of elements of G of order dividing k, the description of the module $\gamma_k(R)/\gamma_{k+1}(R) \otimes_{\mathbb{Z}} \mathbb{Q}$ is not so easy. However, if

$(|G|, k) = 1, k \geqslant 2$, then for $g \neq 1, g^d \neq 1$ for any $d \mid k$, $d \neq 1$. Theorem 4.1 shows that in this case $\chi_k(g) = (1/k)\sum_{d|k}\mu(d) = 0$. Thus χ_k is equal to $(m(k)/|G|)\rho_G$, where $m(k) = \operatorname{rank}\gamma_k(R)/\gamma_{k+1}(R)$. We record the following consequence.

THEOREM 4.3. *If $(|G|, k) = 1$, $k \geqslant 2$, then*

$$\gamma_k(R)/\gamma_{k+1}(R) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \underbrace{\mathbb{Q}G \oplus \cdots \oplus \mathbb{Q}G}_{s}$$

as $\mathbb{Q}G$-modules, where $s = m(k)/|G|$.

In the general situation we are able to prove the following extension of Gaschütz's result.

THEOREM 4.4. *The character χ_k contains the regular character ρ_G of G as a constituent with multiplicity m_k, where*

$$m_2 \geqslant (j-1) + \binom{j-1}{2}|G| + \tfrac{1}{2}|\{g \in G \mid g^2 \neq 1\}|,$$

and

$$m_k \geqslant \frac{1}{k|G|}(\chi(1)^k - \chi(1)^{[k/2]+1}), \qquad k \geqslant 3,$$

where $\chi(1) = 1 + (j-1)|G|$.

Proof. By Lemma 4.1 we may write

$$\chi_k = \frac{1}{k}(\chi^k - \psi_k), \qquad (k \geqslant 2),$$

where

$$\psi_k(g) = -\sum_{\substack{d|k\\d>1}} \mu(d)(\chi(g^d))^{k/d} \qquad (g \in G)$$

is a character of G. For any irreducible character ϕ of G, we first compute the multiplicity of ϕ in ψ_k, $k \geqslant 2$. We have

$$\begin{aligned}\langle\psi_k, \phi\rangle &= -\frac{1}{|G|}\sum_{\substack{d|k\\d>1}}\mu(d)\sum_{g\in G}(\chi(g^d))^{k/d}\overline{\phi(g)}\\ &= -\frac{1}{|G|}\sum_{\substack{d|k\\d>1}}\mu(d)\left\{\sum_{\substack{g\in G\\g^d=1}}\chi(1)^{k/d}\overline{\phi(g)} + \sum_{\substack{g\in G\\g^d\neq 1}}\overline{\phi(g)}\right\}\\ &= -\frac{1}{|G|}\sum_{\substack{d|k\\d>1}}\mu(d)\left\{\sum_{\substack{g\in G\\g^d=1}}(\chi(1)^{k/d} - 1)\overline{\phi(g)} + \sum_{g\in G}\overline{\phi(g)}\right\}\end{aligned}$$

Now

$$\sum_{g\in G} \overline{\phi(g)} = | G |\langle 1_G , \phi\rangle = \begin{cases} 0 & \text{if } \phi \neq 1_G \\ | G | & \text{if } \phi = 1_G ; \end{cases}$$

and

$$\sum_{d|k} \mu(d) = 0 \qquad \text{for} \quad k \geqslant 2.$$

Hence, we now have

$$\langle \psi_k , \phi\rangle = -\frac{1}{| G |} \sum_{\substack{d|k\\ d>1}} \mu(d) \sum_{\substack{g\in G\\ g^d=1}} (\chi(1)^{k/d} - 1) \overline{\phi(g)} + s,$$

where

$$s = \begin{cases} 0 & \text{if } \phi \neq 1_G \\ 1 & \text{if } \phi = 1_G . \end{cases}$$

Taking absolute values we thus get

$$\langle \psi_k , \phi\rangle \leqslant \frac{\phi(1)}{| G |} \sum_{\substack{d|k\\ d>1}} (\chi(1)^{k/d} - 1)\, \nu_d(G) + s,$$

where $\nu_d(G) = |\{x \in G \mid x^d = 1\}|$, and equality holds if k is prime and $\phi = 1_G$. Since ρ_G contains a given irreducible character $\phi(1)$ times, it follows that ψ_k contains ρ_G at most p_k times, where

$$p_k = \frac{1}{| G |} \sum_{\substack{d|k\\ d>1}} (\chi(1)^{k/d} - 1)\, \nu_d(G).$$

Furthermore, since $\chi^k(1) = 1_G(1) + (\chi^k(1) - 1/| G |)\, \rho_G(1)$ and $\chi^k(g) = 1_G(g) + (\chi^k(1) - 1/| G |)\, \rho_G(g)$ for $g \neq 1$, we have $\chi^k = 1_G + (\chi^k(1) - 1/| G |)\, \rho_G$. Thus the multiplicity m_k of ρ_G in χ_k is given by

$$m_k \geqslant \frac{1}{k}\left(\frac{\chi(1)^k - 1}{| G |} - p_k\right).$$

For $k = 2$, we have

$$p_2 = \frac{1}{| G |} (\chi(1) - 1)\, \nu_2(G)$$

so that

$$\begin{aligned} m_2 &\geqslant \frac{1}{2\, | G |} \{(\chi(1)^2 - 1) - (\chi(1) - 1)\, \nu_2(G)\} \\ &= \tfrac{1}{2}\{(j - 1)^2\, | G | + 2(j - 1) - (j - 1)\, \nu_2(G)\} \\ &\geqslant (j - 1) + \binom{j-1}{2} | G | + \tfrac{1}{2}\, |\{g \in G \mid g^2 \neq 1\}|. \end{aligned}$$

For $k \geqslant 3$, we have

$$\begin{aligned} p_k &\leqslant \frac{1}{|G|} \sum_{\substack{d \mid k \\ d>1}} (\chi(1)^{k/d} - 1) \mid G \mid \\ &= \sum_{\substack{d \mid k \\ d>1}} (\chi(1)^{k/d} - 1) \\ &\leqslant \chi(1) + \cdots + \chi(1)^{[k/2]} \\ &\leqslant \chi(1)^{[k/2]+1} - 1/\chi(1) - 1 \\ &\leqslant \chi(1)^{[k/2]+1} - 1/\mid G \mid, \end{aligned}$$

since $\chi(1) = 1 + (j-1) \mid G \mid$, so that

$$m_k \geqslant \frac{1}{k \mid G \mid} (\chi(1)^k - \chi(1)^{[k/2]+1})$$

as required.

As an immediate consequence of Theorem 4.4 we obtain the following result (see Mittal and Passi [8]).

COROLLARY 4.5. *For all $k \geqslant 1$, $\gamma_k(R)/\gamma_{k+1}(R)$ is a faithful $\mathbb{Z}(F/R)$-module for F/R finite.*

Remark. A direct proof that χ_k contains at least one regular constituent may be obtained as follows. Since

$$R/R' \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q} \oplus \underbrace{\mathbb{Q}G \oplus \cdots \oplus \mathbb{Q}G}_{\longleftarrow j-1 \longrightarrow}$$

and $j > 1$, we can choose $r \in R$ such that the span of $\{r^g R' \mid g \in G\}$ is a $\mathbb{Q}G$-submodule which affords the regular character ρ_G of G. Let $u \in R \backslash R'$ be such that uR' is fixed by G and that it is independent of $\prod_{g \in G} r^g R'$ (this is possible since C_1/R' has rank at least two). Then the $\mathbb{Q}$-span of

$$\{[r^g, \underbrace{u, \ldots, u}_{\leftarrow k-1 \rightarrow}]\, \gamma_{k+1}(R) \mid g \in G\}$$

is a $\mathbb{Q}G$-submodule of $\gamma_k(R)/\gamma_{k+1}(R) \otimes_{\mathbb{Z}} \mathbb{Q}$ which affords the regular character ρ_G.

5. THE RANK OF THE CENTRE OF $F/\gamma_{k+1}(R)$

Let $C_k/\gamma_{k+1}(R)$ $(k \geqslant 1)$ denote the set of fixed points of $G = F/R$ on $\gamma_k(R)/\gamma_{k+1}(R)$. Then $C_k/\gamma_{k+1}(R)$ is precisely the centre of $F/\gamma_{k+1}(R)$ (see for instance [5]). Lemma 2.2 (with A replaced by $\gamma_k(R)/\gamma_{k+1}(R)$ shows that the rank r_k of $C_k/\gamma_{k+1}(R)$ is given by

$$r_k = \langle \chi_k , 1_G \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_k(g),$$

where χ_k is the character of the k-th relation module $\gamma_k(R)/\gamma_{k+1}(R)$. Theorems 4.1 and 4.2 now show that

$$\begin{aligned} r_k &= \frac{1}{k\,|G|} \sum_{d|k} \mu(d) \sum_{g \in G} (\chi(g^d))^{k/d} \\ &= \frac{1}{k\,|G|} \sum_{d|k} \mu(d) \left\{ \sum_{\substack{g \in G \\ g^d=1}} [(\chi(1))^{k/d} - 1] + \sum_{g \in G} 1 \right\} \\ &= \frac{1}{k\,|G|} \sum_{d|k} \mu(d) \{\nu_d(G)(m^{k/d} - 1) + |G|\}, \end{aligned}$$

where $\nu_d(G) = |\{g \in G \mid g^d = 1\}|$ and $m = \chi(1) = \operatorname{rank} R$. When $k = 1$, we obtain

$$\begin{aligned} r_1 &= \frac{1}{|G|}(m - 1 + |G|) \\ &= \frac{1}{|G|}(1 + |G|(j-1) - 1 + |G|) \\ &= j \ (= \operatorname{rank} F); \end{aligned}$$

when $k > 1$, we obtain

$$r_k = \frac{1}{k\,|G|} \sum_{d|k} \mu(d)\, \nu_d(G)(m^{k/d} - 1),$$

since $\sum_{d|k} \mu(d) = 0$.

We have thus proved the following

THEOREM 5.1. *Let $C_k/\gamma_{k+1}(R)$ ($k \geqslant 1$) be the centre of $F/\gamma_{k+1}(R)$. Then the rank r_k of $C_k/\gamma_{k+1}(R)$ is given by the formula*

$$r_1 = j, \qquad \textit{the rank of } F,$$

$$r_k = \frac{1}{k\,|G|} \sum_{d|k} \mu(d)\, \nu_d(G)(m^{k/d} - 1) \qquad (k \geqslant 2),$$

where $m = 1 + (j - 1)\,|G| =$ rank of R and $\nu_d(G)$ is the number of elements of G whose order divides d.

Since $m \geqslant |G| + 1$ and $\nu_d(G) \leqslant |G|$, the first term $(1/k\,|G|)(m^k - 1)$ of r_k is much larger than the sum of all other terms and hence gives a fairly good estimate for r_k. However, for $(|G|, k) = 1$, $k \geqslant 2$, Theorem 4.3 yields the following result.

THEOREM 5.2. *If* $(|G|, k) = 1$, $k \geqslant 2$, *then the rank* r_k *of* $C_k/\gamma_{k+1}(R)$ *is* $m(k)/|G|$, *where* $m(k)$ *is the rank of* $\gamma_k(R)/\gamma_{k+1}(R)$.

Remark. Our computation of the rank of $C_k/\gamma_{k+1}(R)$ does not lend itself to the actual computation of $C_k/\gamma_{k+1}(R)$. The dependence of r_k on $\nu_k(G)$ makes the task of actual computation of $C_k/\gamma_{k+1}(R)$ a very laborious one except for small groups G and small values of k. However, we note that $C_k/\gamma_{k+1}(R)$ is a $\mathbb{Z}$-direct summand of $\gamma_k(R)/\gamma_{k+1}(R)$. To see this we note that if $(u\gamma_{k+1}(R))^t \in C_k/\gamma_{k+1}(R)$ for some $u \in \gamma_k(R)$ and $t \geqslant 1$, then $[u^t, f] \in \gamma_{k+1}(R)$ for all $f \in F$. But $[u^t, f] \equiv [u, f]^t \bmod \gamma_{k+1}(R)$ and $\gamma_k(R)/\gamma_{k+1}(R)$ is free abelian. Hence $[u, f] \in \gamma_{k+1}(R)$ which implies $u\gamma_{k+1}(R) \in C_k/\gamma_{k+1}(R)$. We next note that $C_k/\gamma_{k+1}(R) \otimes_{\mathbb{Z}} \mathbb{Q} = D_k \otimes_{\mathbb{Z}} \mathbb{Q}$, where

$$D_k = \left\{ \sum_{g \in G} b \cdot g \mid b \in \gamma_k(R)/\gamma_{k+1}(R) \right\}.$$

Using standard techniques from the abelian group theory we can compute a basis $b_1, \ldots, b_r$ of $\gamma_k(R)/\gamma_{k+1}(R)$, such that $h_1 b_1, \ldots, h_s b_s$ ($h_i > 0$, $s \leqslant r$) is a basis for D_k. Then $C_k/\gamma_{k+1}(R)$ is the group generated by $b_1, \ldots, b_s$.

6. SOME GENERAL PROPERTIES OF $F/\gamma_{k+1}(R)$

In this section we use the results of Section 2 to obtain some further information concerning the groups $F/\gamma_{k+1}(R)$, where $G = F/R$ is finite. Let $C_k/\gamma_{k+1}(R)$ denote the centre of $F/\gamma_{k+1}(R)$. Then since $C_k/\gamma_{k+1}(R)$ is the set of fixed points of G on $\gamma_k(R)/\gamma_{k+1}(R)$, Lemma 2.1 yields the following useful results.

LEMMA 6.1. $C_k \cap [\gamma_k(R), F] \leqslant \gamma_{k+1}(R)$ *for all* $k \geqslant 1$.

LEMMA 6.2. *If* F/R *is finite, then*

$$[\gamma_k(R), F]/[\gamma_k(R), \underbrace{F, \ldots, F}_{s}]\, \gamma_{k+1}(R)$$

is finite of exponent dividing $|G|^{s-1}$ *for all* $s \geqslant 2$ *and* $k \geqslant 1$.

[*To obtain Lemma* 6.2, *note that* $[\gamma_k(R), F]/\gamma_{k+1}(R)$ *and*

$$[\gamma_k(R), \underbrace{F, \ldots, F}_{s}]\, \gamma_{k+1}(R)/\gamma_{k+1}(R)$$

are free abelian groups of the same rank].

We next obtain the following generalization of a result of Gruenberg [4] for the case $k = 1$.

THEOREM 6.3. *If F/R is a non-trivial finite group, then, for each $k \geqslant 1$, the second centre of $F/\gamma_{k+1}(R)$ coincides with the centre of $F/\gamma_{k+1}(R)$.*

Proof. Let $C_k^*/\gamma_{k+1}(R)$ denote the second centre of $F/\gamma_{k+1}(R)$ for each $k \geqslant 1$. If $k = 1$, then Gruenberg's result [4] gives $C_1^* = C_1$. Alternatively, $[C_1^*, F, F] \leqslant R'$, so $[C_1^*, F] \leqslant C_1$. Hence, $[C_1^*, R] \leqslant C_1 \cap [F, R] \leqslant R'$ by Lemma 6.1. Hence, $C_1^* \leqslant R$ since F/R acts faithfully on R/R'. But then $[C_1^*, F] \leqslant [R, F] \cap C_1 \leqslant R'$ and it follows that $C_1^* \leqslant C_1$.

Suppose now that $k > 1$. Then $[C_k^*, F, F] \leqslant \gamma_{k+1}(R)$ implies that $[C_k^*, F] \leqslant C_k \leqslant \gamma_k(R)$ and hence, $C_k^* \leqslant C_{k-1} \leqslant \gamma_{k-1}(R)$. Now $[R, F] \neq R'$ since $R \neq F$ and $[C_k^*, [R, F]] \leqslant [C_k^*, F, F] \leqslant \gamma_{k+1}(R)$. By Theorem 5.10 of [7], it follows that $C_k^* \leqslant \gamma_k(R)$. Hence, $[C_k^*, F] \leqslant C_k \cap [\gamma_k(R), F] \leqslant \gamma_{k+1}(R)$, by Lemma 6.1. This proves $C_k^* \leqslant C_k$ and hence, $C_k^* = C_k$.

As a further application, we give an alternative proof of the following theorem for F/R finite.

THEOREM 6.4 (Gupta and Gupta [5]). *For each $k \geqslant 2$, the centre of $F/[\gamma_k(R), F]$ is precisely $\gamma_k(R)/[\gamma_k(R), F]$.*

Proof. Let $w \in F$ with $[w, F] \leqslant [\gamma_k(R), F]$. Then we want to show that $w \in \gamma_k(R)$. We first observe that since $[w, F] \leqslant \gamma_k(R)$, $w \in C_{k-1} \leqslant R$ (since $k \geqslant 2$). Suppose $w \notin \gamma_k(R)$. If $u \in C_1 \backslash \gamma_2(R)$, then by Theorem 5.12 of [7], $[w, u] \notin \gamma_{k+1}(R)$ unless $k = 2$ and $w\gamma_2(R)$, $u\gamma_2(R)$ are $\mathbb{Q}$-dependent in $\gamma_1(R)/\gamma_2(R) \otimes_{\mathbb{Z}} \mathbb{Q}$. Since $C_1/\gamma_2(R)$ has rank at least 2 we may therefore (also in the case $k = 2$) choose u in C_1 so that $[w, u] \notin \gamma_{k+1}(R)$. But $[w, u, F] \leqslant [w, F, u]$ $[w, [u, F]] \leqslant \gamma_{k+1}(R)$ implies that $[w, u] \in C_k$. Since, by hypothesis, $[w, F] \leqslant [\gamma_k(R), F]$ it follows that $[w, u] \in C_k \cap [\gamma_k(R), F] \leqslant \gamma_{k+1}(R)$, by Lemma 6.1. This gives the required contradiction.

7. EMBEDDING OF THE FREE LIE ALGEBRAS

Let K be a field of characteristic zero and A a free K-module on $a_1, \ldots, a_n$ $(n < \infty)$. We form the tensor space $T(A)$ which may be regarded as a graded associative K-algebra on A. Let $L(A)$ denote the graded Lie algebra generated by $a_1, \ldots, a_n$. For each $k \geqslant 1$, let $T^k(A)$, $L^k(A)$ denote the homogeneous component of degree k of $T(A)$ and $L(A)$ respectively. Then $L^k(A)$ is naturally embedded in $T^k(A)$ via $[a, b] = a \otimes b - b \otimes a$. For $k \geqslant 2$, define

$$\Omega_k = [e - (12)] \cdots [e - (12 \cdots k)],$$

where $(12 \cdots j)$ is the cyclic permutation of $\{1, 2, \ldots, j\}$ and e is the identity element of $\mathbb{Z}S_k$, S_k being the symmetric group on $\{1,\ldots, k\}$. We refer the reader to Section 5.9 of [7] where it is established that

$$L^k(A) = T^k(A) \cdot \Omega_k \qquad \text{and} \qquad \Omega_k{}^2 = k\Omega_k .$$

Since as K-modules $T^k(A) \cdot \Omega_k = T^k(A) \cdot \epsilon_k$, where $\epsilon_k = (1/k)\,\Omega_k$ is an idempotent, we have

$$L^k(A) = T^k(A) \cdot \epsilon_k \qquad (\epsilon_k{}^2 = \epsilon_k).$$

If G is a finite group acting on A as a group of transformations and χ is the character of G on A then clearly χ^k is the character of G on $T^k(A)$. The Brandt formula discussed in Section 3 is that of describing the character χ_k of the representation induced by G on $L^k(A)$ in terms of χ. Since the characters on $T^k(A)$ are easy to compute, the above embedding suggests an alternative approach to obtain a formula for χ_k. Indeed, if T is a linear transformation of A and T_k is the corresponding transformation induced by T on $L^k(A)$, then we have

$$\operatorname{tr} T_k = \operatorname{tr}\left[\overset{k}{\bigotimes} T \cdot \epsilon_k\right],$$

where

$$\overset{k}{\bigotimes} T = \underbrace{T \otimes \cdots \otimes T}_{\longleftarrow\, k \,\longrightarrow}.$$

If σ_i, σ_j are conjugate in S_k, then

$$\operatorname{tr}\left[\overset{k}{\bigotimes} T \cdot \sigma_i\right] = \operatorname{tr}\left[\overset{k}{\bigotimes} T \cdot \sigma_j\right].$$

so that the trace formula takes the form

$$\operatorname{tr} T_k = \operatorname{tr}\left[\overset{k}{\bigotimes} T \cdot \epsilon_k^*\right], \tag{$**$}$$

where ϵ_k^* is obtained from ϵ_k by identifying all conjugate elements in the expansion of Ω_k. For example, if $k = 4$ then

$$\Omega_4 = e - (12) + (13) - (123) - (1234) + (134) - (14)(23) + (1324)$$

and

$$\Omega_4^* = e - (14)(23) = e - (12)(34), \qquad \text{etc.}$$

As in Section 2, we may assume that T is diagonalizable over K. Then, if $\sigma = \zeta_1 \cdots \zeta_q \in S_k$ is a product of disjoint cycles with length of $\zeta_i = h_i$, we have

$$\operatorname{tr}\left[\overset{k}{\bigotimes} T \cdot \sigma\right] = (\operatorname{tr} T^{h_1}) \cdots (\operatorname{tr} T^{h_q}).$$

Thus substituting into (**) we can get a formula for tr T_k. However, the difficulty in this approach lies in determining ϵ_k^*. The direct calculations yield information only for small values of k. Instead, we can use the trace formula of Theorem 3.1 to derive in an elementary way, the following information on ϵ_k^* due to Wever [12].

THEOREM 7.1 (Wever). *For $k \geqslant 2$, $\epsilon_k^* = (1/k)\sum_{d|k}\mu(d)p(d)$, where $p(d) \in S_k$ is any product of k/d disjoint d cycles.*

Proof. Let A be a free abelian group on $a_1,\dots,a_n$. Define a map T on $A \otimes_{\mathbf{Z}} \mathbb{Q}[\lambda_1,\dots,\lambda_n]$ by $Ta_i = \lambda_i a_i$ and extend it by linearity ($\lambda_1,\dots,\lambda_n$ are commuting indeterminates). For each partition $\mathbf{h} = \{h_1,\dots,h_q\}$ of k with $q \geqslant 1$ and $h_1 \geqslant \cdots \geqslant h_q \geqslant 1$, put

$$f(\mathbf{h}) = f(h_1)\cdots f(h_q),$$

where $f(r) = \lambda_1{}^r + \cdots + \lambda_n{}^r$. For $n \geqslant k$, the set of all $f(\mathbf{h})$ over all such partitions $\mathbf{h}$ of k is linearly independent over $\mathbb{Q}$ since $f(1),\dots,f(k)$ are algebraically independent over $\mathbb{Q}$ (see for instance [11], page 158). Let

$$\epsilon_k^* = \sum_{i=1}^{s} n_i \zeta_{i(1)} \cdots \zeta_{i(q(i))},$$

where s is the number of conjugacy classes of S_k, $n_i \in \mathbb{Q}$ and $\zeta_{i(1)} \cdots \zeta_{i(q(i))}$ is a product of disjoint cycles corresponding to the partition $\mathbf{h}_i = \{h_{i(1)},\dots,h_{i(q(i))}\}$ of k. Then from (**) we obtain

$$\begin{aligned}
\operatorname{tr} T_k &= \sum_{i=1}^{s} n_i \operatorname{tr}\left[\bigotimes^{k} T \cdot \zeta_{i(1)} \cdots \zeta_{i(q(i))}\right] \\
&= \sum_{i=1}^{s} n_i (\operatorname{tr} T^{h_{i(1)}}) \cdots (\operatorname{tr} T^{h_{i(q(i))}}) \\
&= \sum_{i=1}^{s} n_i (\lambda_1^{h_{i(1)}} + \cdots + \lambda_n^{h_{i(1)}}) \cdots (\lambda_1^{h_{i(q(i))}} + \cdots + \lambda_n^{h_{i(q(i))}}) \\
&= \sum_{i=1}^{s} n_i f(h_{i(1)}) \cdots f(h_{i(q(i))}) \\
&= \sum_{i=1}^{s} n_i f(\mathbf{h}_i).
\end{aligned}$$

On the other hand, by Lemma 3.1,

$$\operatorname{tr} T_k = \frac{1}{k} \sum_{d|k} \mu(d) f(\mathbf{h}(\mathbf{d})),$$

where $\mathbf{h}(\mathbf{d}) = \{d, \ldots, d\}$ is the partition of k (d repeats k/d times). A comparison of $\sum_{i=1}^{s} n_i f(\mathbf{h}_i)$ with $(1/k) \sum_{d|k} \mu(d) f(\mathbf{h}(\mathbf{d}))$ now shows that $n_i = 0$ unless $\mathbf{h}_i = \mathbf{h}(\mathbf{d})$ for some d dividing k and $n_i = (1/k)\,\mu(d)$. Thus $\epsilon_k^* = (1/k) \sum_{d|k} \mu(d)\, p(d)$, as required.

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